STATISTICS IN TRANSITION new series, December 2021
Vol. 22, No. 4, pp. 59–76, DOI 10.21307/stattrans-2021-038
Received – 02.06.2020; accepted – 11.02.2021

Agu-Eghwerido distribution, regression model and applications

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ABSTRACT

Modelling lifetime data with simple mathematical representations and an ease in obtaining the parameter estimate of survival models are crucial quests pursued by survival researchers. In this paper, we derived and introduced a one-parameter distribution called the Agu-Eghwerido (AGUE) distribution with its simple mathematical representation. The regression model of the AGUE distribution was also presented. Several basic properties of the new distribution, such as reliability measures, mean residual function, median, moment generating function, skewness, kurtosis, coefficient of variation, and index of dispersion, were derived. The estimation of the proposed distribution parameter was based on the maximum likelihood estimation method. The real-life applications of the distribution were illustrated using two real lifetime negatively and positively skewed data sets. The new distribution provides a better fit than the Pranav, exponential, and Lindley distributions for the data sets. The simulation results showed that the increase in parameter values decreases the mean squared error value. Similarly, the mean estimate tends towards the true parameter value as the sample sizes increase.

Key words: AGUE distribution, AGUE regression model, moment generating function, means residual function, hazard rate function, survival rate function.

1. Introduction

Introducing one-parameter distributions is a continuous concern for distribution theory and survival researchers. Thus, the researchers desire to introduce mathematically tractable and flexible lifetime probability models which can represent the random behaviour of real-world lifetime situations without difficulty.

In the statistical literature, many one-parameter, as well as two or more parameters probabilistic models have been introduced by researchers for modelling lifetime situations. Some of these probability models provide good results for various life situations. However, some of these developed models do not give good results for real-life scenarios. This might be the case arising from physical sciences, medical sciences, biological sciences, agricultural science, engineering among others. Similarly, some of these models require quite a complex and time-consuming algorithm for their parameter estimation.

Lindley (1958) introduced a parameter Lindley distribution. Ghitany, Atieh, and Nadarajah (2008) applied the mathematical treatment to a parameter Lindley distribution. Shanker

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(2015a, 2016) proposed one-parameter Akash and Aradhana distributions. Shanker and Shukla (2017) proposed one-parameter Ishita distribution. Shanker (2016) proposed one-parameter Sujatha distribution. Shukla (2018) proposed one-parameter Pranav distribution. Odom and Ijomah (2019) proposed one-parameter Odoma distribution. However, their regression models were not explored. These distributions yielded good fit over Lindley and exponential distributions for various data sets. Furthermore, the Pranav distribution by Shukla was applied on three real lifetime data sets and provides a better fit to the data sets than Lindley, Sujatha, Akash, and Ishita distributions respectively. However, the Odoma distribution was applied to strength data of glass of the aircraft window and provides a better fit than Pranav, Sujatha, Aradhana, Akash, Lindley, and exponential distributions respectively. The mathematical properties of these probability models, as well as parameter estimation, has been studied. Most of these probability models provide a good fit in some real-life situations and performed poorly in others. Due to the shortfall of some of these models, for instance the constant hazard rate of the exponential distribution could limit its wider applications to increasing hazard rate lifetime situations. Hence, some of these distributions have been extended, generalized, and applied in the literature by the addition of an extra parameter by researchers including Zakerzadeh and Dolati (2009) generalized Lindley distribution. Nadarajah et al. (2011) extended exponential distribution. Gómez and Calderín (2011) derived the discrete Lindley distribution, properties, and applications. Bakouch et al. (2012) obtained exponentiated exponential binomial distribution. Shanker and Mishra (2013) obtained a sized-biased Quasi Poisson-Lindley distribution. Agu and Onwukwe (2019) proposed exponentiated Laplace distribution as the extension of the Laplace distribution. Eghwerido et al. (2020) proposed the alpha power Gompertz distribution. Shanker and Amanuel (2013) obtained a new quasi Lindley distribution. Ghitany et al. (2013) proposed the Power Lindley distribution. Merovci (2013) extends the Rayleigh distribution. Agu and Runyi (2018) studied the goodness of fit tests for normal distribution. Warahena and Pararai (2014) proposed the generalized power Lindley distribution. Oluyede and Yang (2014) generalized the inverse Weibull distribution among others. However, the extensions do not guarantee high confidence in the model reliability. These extensions could lead to complex mathematical representation, complex and time-consuming algorithm, and difficulty in the estimation of parameters resulting in unreliable results.

In this respect, this paper is motivated to introduce a heavy-tailed simple mathematically structured one-parameter probability model with non-decreasing survival and hazard functions, derive its regression model, require less time and simple algorithm, and ease in the parameter estimation based on the test statistics performance results obtained from real-life scenarios that are more reliable. The proposed model was generated by making use of gamma and exponential distributions because of their inherent uniform base and constant failure rate respectively. However, in statistical modelling, the Weibull, Gompertz, exponential and Lindley models are very popularly used in modelling compared to the lognormal and gamma distributions. This is as a result of the inability to express their survival functions in a closed form. Although, the exponential function has one-parameter with a constant hazard rate, the Lindley has one-parameter monotonic decreasing hazard rate with a mixture component of a gamma model with a shape parameter 2 and exponential distribution. Its mixture proportions are $\frac{\lambda}{\lambda+1}$ and $\frac{1}{\lambda+1}$ for a scale parameter λ. Also, the Akash model tries to improve the Lindley distribution by using a mixture of exponential and gamma with a shape parameter 3 and mixing proportions $\frac{\lambda^2}{\lambda^2+2}$ and $\frac{2}{\lambda^2+1}$. Thus, to improve the flexibility of the Lindley, Pranav, exponential, and Akash distributions, this article introduces a parsimonious tractable distribution model called the AGUE distribution.

Let us consider two components mixture of one-parameter gamma distribution having shape parameter 3 and scale parameter λ and exponential distribution with scale parameter λ with their mixing proportions $\frac{\lambda^6}{\lambda^6+8}$ and $\frac{8}{\lambda^6+8}$ respectively. This idea would be used to develop the one-parameter distribution with simple mathematical representation.

The rest of the paper is structured as follows: Section 2 introduced the new distribution. Section 3 explored the reliability measures of the new distribution. Section 4 explored the moment generating function for the new distribution. Section 5 discussed the parameter estimation for the new distribution. Section 5.1 introduced the regression model for the new distribution. In Section 6, two real lifetime data set were adopted to illustrate the behaviour of the new distribution. Section 7 provided the concluding remarks.

2. The AGUE distribution

This section introduces the proposed model. It also, examines some potential properties. Let x be a random variable. Then, the new lifetime density function is introduced as

$$P(x) = \frac{\lambda^3}{\lambda^6+8}\left[\lambda^4+4x^2\right]exp(-\lambda x), \tag{1}$$

where $\lambda > 0,\ x > 0$. We would call the probability density function (pdf) of the one-parameter life time distribution in (1) "AGU-EGHWERIDO (AGUE) distribution". See the proof of the density function in Appendix. It follows from (1) that

$$\frac{dP(x)}{dx} = \frac{\lambda^3}{\lambda^6+8}\left[8x-\lambda^5-4\lambda x^2\right]exp(-\lambda x).$$

For

1. $\lambda < 1$, $\frac{dP(x)}{dx} = 0$, this implies that $P(x)$ is maximized at x_0.
2. $\lambda \geq 1$, $\frac{dP(x)}{dx} \leq 0$, this implies that $P(x)$ is decreasing at x.

The cumulative distribution function (cdf) of the AGUE distribution in (1) is given as

$$\frac{dP(x)}{dx} = 1-\frac{1}{\lambda^6+8}\left[\lambda^6+4\lambda^2x^2+8(\lambda x+1)\right]exp(-\lambda x), \tag{2}$$

where $\lambda > 0, x > 0$.

3. Reliability measures

In this section, some reliability properties of the AGUE distribution were examined and investigated.

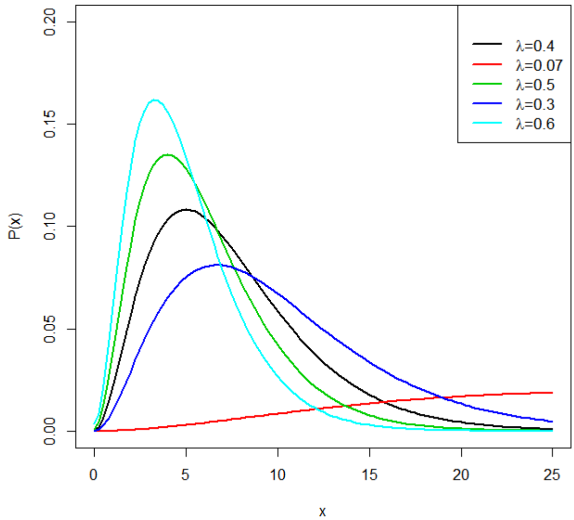

Figure 1: Pdf of AGUE distribution with different parameter cases

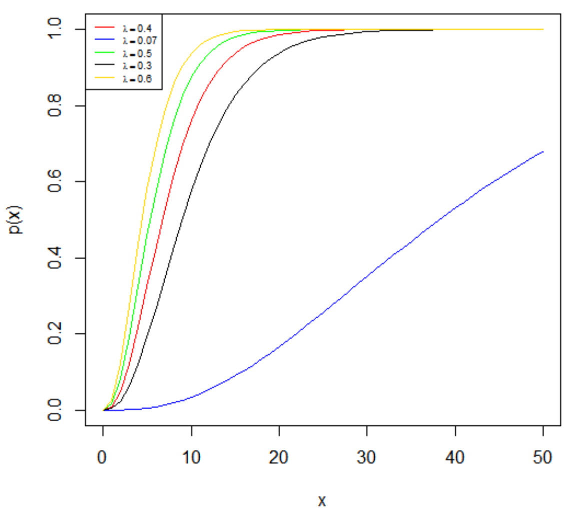

Figure 2: Cdf plot of the AGUE distribution with different parameter cases

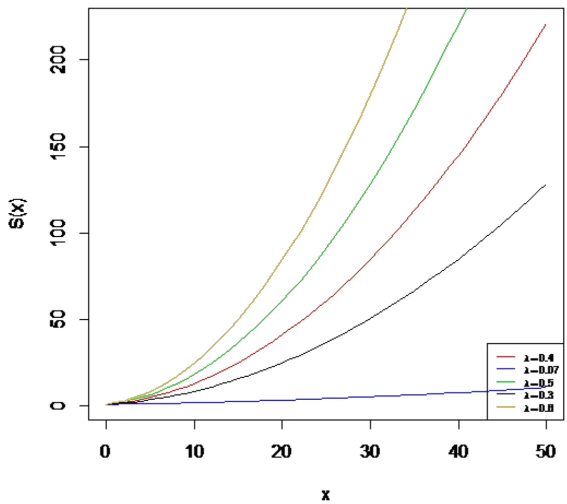

Figure 3: Survival plot of the AGUE distribution with different parameter cases

Let X be a continuous random variable with pdf $P(x)$ and cdf $p(x)$. Then, the survival rate function is expressed as

$$S(x) = 1 - \frac{1}{\lambda^6 + 8}\left[\lambda^6 + 4\lambda^2 x^2 + 8(\lambda x + 1)\right] exp(-\lambda x). \tag{3}$$

The survival rate plot for different parameter values is shown in Figure 3.

3.1. The failure rate function

The failure rate function is given as

$$H(x) = \lim_{\triangle x \longrightarrow 0} \frac{v(X < x + \triangle x | R > x)}{\triangle x} = \frac{P(x;\lambda)}{S(x;\lambda)}.$$

Thus, we have the AGUE failure rate function as

$$H(x) = \frac{\lambda^3\left[\lambda^4 + 4x^2\right]}{\lambda^6 + 4\lambda^2 x^2 + 8(\lambda x + 1)}. \tag{4}$$

However, for

1. $H(0) = \frac{\lambda^7}{\lambda^6 + 8}$.

2. $H(x)$ is an increasing function in x, λ and $\frac{\lambda^7}{\lambda^6+8} < H(x) < \lambda$.

The hazard rate plot for different parameter values is shown in Figure 4.

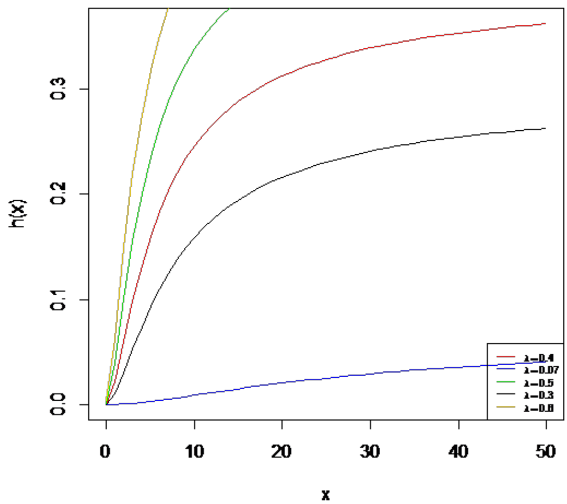

Figure 4: Hazard plot of the AGUE distribution with different parameter cases

3.2. The mean residual function

Let be a random variable with pdf and cdf defined in (1) and (9) respectively. Thus, the mean residual function of X is defined as

$$m(x) = E[X - x|X > x] = \frac{1}{p(x)} \lim_{c\to\infty} \int_x^c [1 - p(t)]dt.$$

However, for $1 - p(x) > 0$.

$$m(x) = \frac{\lambda^5 + 4\lambda x^2 + 16x + \frac{24}{\lambda}}{\lambda^6 + 4\lambda^2 x^2 + 8(\lambda x + 1)}, \tag{5}$$

Moreover, for

1. $m(0) = \frac{\lambda^5 + \frac{24}{\lambda}}{\lambda^6 + 8}$.
2. $m(x)$ is a decreasing function in x and λ.

The mean residual plot for different parameter values is shown in Figure 5.

3.3. The median

The median of a random variable X is expressed as

$$m_2(x) = \lim_{c\to\infty} \int_0^c |x - K|P(x)dx,$$

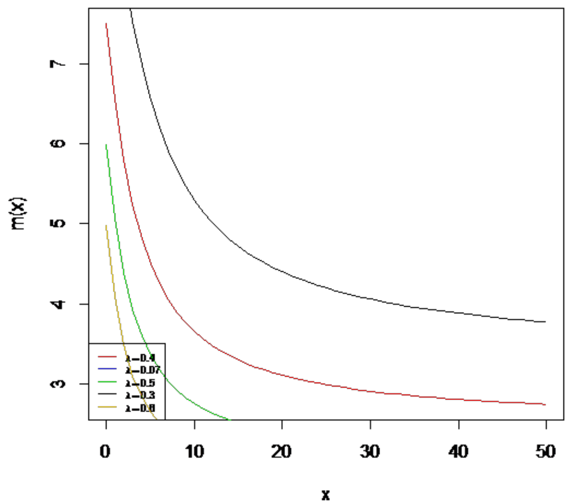

Figure 5: Mean residual plot of the AGUE distribution with different parameter cases

where $\mu = E(X)$ and K = Median (X). The above measures can be calculated using the relationship defined as

$$E(|X-k|) = \int_0^k (k-x)P(x)dx + \lim_{c\to\infty}(x-k)P(x)dx = 2\int_0^k (k-x)P(x)dx. \tag{6}$$

Thus, the median of the AGUE distribution can be given as

$$m_2(x) = 2\Bigg[K - \frac{\lambda^5 + \frac{24}{\lambda}}{\lambda^6+8} - \frac{1}{\lambda^6+8}\Bigg[exp(-\lambda K)(4\lambda K^2 \\ +16K + \lambda^5 + \frac{24}{\lambda}) + \lambda^6 K + 8K - \lambda^5 - \frac{24}{\lambda}\Bigg]\Bigg].$$

3.4. The quantile function

The quantile function of the AGUE distribution is obtained using the Lambert W function with W function defined as the solution to the equation $W(x)exp(W(x)) = x \in [-1,\infty)$, where W_0 is the principal branch of the Lambert function. The solution of the Lambert W function of $W(x)exp(W(x)) = x$, with $W_0(0) = 0$ and $W_0(x)$ increases with increase in x. Thus, for $x > 0$, $u^* = (\lambda^6+8)(1-u)(\lambda^6+4x^2\lambda^2+(9\lambda x+8))$, $\lambda < 0$, and $u \in (0,1)$. Thus, the $x_u = W_0(exp(u^*)) - \frac{\lambda^3-2}{\lambda}$.

A simulation is obtained using the quantile function obtained above. The values of the parameter are chosen as 0.5, 1.0 and 2.5 for sample size 5, 10, 50, 100, 150, 250, 350 and 500. The sample size is replicated 5000 times. Table 1 shows the results.

Table 1. Simulation results for AGUE distribution

n	Parameter	Average Estimate	Bias	MSE
5	λ =0.5	0.9950	0.0150	0.1929
	λ=1.0	1.2008	0.2008	0.5558
	λ=2.5	2.7462	0.5349	0.5071
10	λ =0.5	0.8516	0.0847	0.1077
	λ =1.0	1.1847	0.1838	0.4671
	λ =2.5	2.6838	0.1921	0.4014
50	λ =0.5	0.6664	0.0972	0.1005
	λ =1.0	1.0072	0.1583	0.3273
	λ =2.5	2.5949	0.1049	0.2702
100	λ =0.5	0.4754	0.1625	0.0071
	λ =1.0	1.0025	0.1348	0.1929
	λ =2.5	2.4712	0.0881	0.1052
150	λ =0.5	0.4544	0.1109	0.0045
	λ =1.0	1.0009	0.0181	0.0824
	λ =2.5	2.4519	0.0324	0.0841
250	λ =0.5	0.5013	0.2733	0.0024
	λ =1.0	1.0008	0.0112	0.0053
	λ =2.5	2.5030	0.0030	0.0070
350	λ =0.5	0.5002	0.4814	0.0015
	λ =1.0	1.0002	0.0088	0.0034
	λ =2.5	2.5003	0.0018	0.0055
500	λ =0.5	0.5001	0.4948	0.0011
	λ =1.0	1.0005	0.3651	0.0041
	λ =2.5	2.5001	0.2476	0.0006

In Table 1, increase in parameter values decreases the MSE. Also, the mean estimate tends to the true parameter value as the sample sizes increase.

4. Moment generating function

The AGUE moment generating function is expressed as

$$\begin{aligned} m_R(t) &= \frac{\lambda^3}{\lambda^6+8}\lim_{c\to\infty}\left[\lambda^6\sum_{m=0}^{c}(\frac{t}{\lambda})^m+\frac{8}{\lambda}\sum_{m=0}^{c}\binom{m+2}{m}(\frac{t}{m})^m\right] \\ &= \lim_{c\to\infty}\left[\frac{t^m\left[\lambda^6+(m+2)(m+4)\right]}{\lambda^m(\lambda^6+8)}\right]. \end{aligned} \tag{7}$$

Thekth moment about origin μ_k' obtained as coefficient of $\frac{t^k}{k!}$ in $M_X(t)$ of the AGUE distribution is given by

$$\mu_k' = E(X^k) = \frac{k!\left[\lambda^6 + (k+2)(k+4)\right]}{\lambda^k(\lambda^6+8)} \quad k = 1,2,3,\cdots \tag{8}$$

In particular, we can obtain the first four moments of the AGUE distribution as

$$\mu_1' = \frac{\lambda^6+15}{\lambda(\lambda^6+18)},\ \mu_2' = \frac{2(\lambda^6+24)}{\lambda^2(\lambda^6+8)},\ \mu_3' = \frac{6(\lambda^6+35)}{\lambda^3(\lambda^6+8)},\ \textit{and}\ \mu_4' = \frac{24(\lambda^6+48)}{\lambda^4(\lambda^6+8)}.$$

Using the relationship between the raw moment and the central moment or the moment about mean, we obtain the central of the AGUE distribution (1) as

$$\mu_k = E(X-\mu)^k = \sum_{m=0}^{k} \binom{k}{m} \mu_m' (-\mu)^{k-m}, \quad m = 0,1,2,\cdots,k.$$

In particular, we have

$$\mu_0 = 1, \quad \mu_1 = 0.$$

$$\mu_2 = \mu_2' - \mu^2 = \frac{\lambda^{12}+34\lambda^6+159}{\lambda^2(\lambda^6+8)^2}.$$

$$\mu_3 = \mu_3' - 3\mu\mu_2' + 2\mu^3 = \frac{2\lambda^{18}+114\lambda^{12}+62\lambda^6+2910}{\lambda^3(\lambda^6+8)^3}.$$

$$\mu_4 = \mu_4' + 6\mu^2\mu_2' - 4\mu\mu_3' - 3\mu^4 = \frac{9\lambda^{24}+1194\lambda^{18}+16002\lambda^{12}+31164\lambda^6+96963}{\lambda^4(\lambda^6+8)^4}.$$

The coefficient of skewness $\sqrt{\beta_1}$, coefficient of kurtosis β_2 , coefficient of variation $C.V$, and index of dispersion γ of the AGUE distribution (1) are obtained as

$$C.V = \frac{\sqrt{\lambda^{12}+34\lambda^6+159}}{\lambda^6+15}, \quad \sqrt{\beta_1} = \frac{2\lambda^{18}+114\lambda^{12}+62\lambda^6+2910}{(\lambda^{12}+34\lambda^6+159)^{\frac{3}{2}}}$$

$$\beta_2 = \frac{9\lambda^{24}+1194\lambda^{18}+16002\lambda^{12}+3116\lambda^6+96963}{(\lambda^{12}+34\lambda^6+159)^2},\ \textit{and}\ \gamma = \frac{\lambda^{12}+34\lambda^6+159}{\lambda(\lambda^6+8)(\lambda^6+15)}.$$

Table 2 shows the variance, mean, skewness, kurtosis, coefficients of variation, and index of dispersion for different parameter values for the AGUE distribution. The results in Table 2 shown that the parameter values increases with decrease in the variance values.

Table 2. Coefficients of variation, mean, variance, skewness, kurtosis and index of dispersion for the AGUE distribution.

(λ)	μ	σ^2	CV	$\sqrt{\beta_1}$	β_2	γ
0.4	4.6864	15.5250	0.8408	1.4545	3.8292	3.1567
0.01	187.500	24843.750	0.8406	1.4514	3.8354	132.33
0.6	3.1165	6.8894	0.8422	1.4410	3.7669	2.0684
0.5	3.7466	9.9319	0.8412	1.4519	3.8110	2.4999
0.016	117.188	9704.59	0.8406	1.4514	3.8354	82.647
2	0.5486	0.3101	1.0151	1.9281	5.9972	4.0703

4.1. Stochastic Orderings

The comparative behaviour of continuous random variables can be evaluated using stochastic ordering.

A random variable X is said to be smaller than a random variable Y (Shaked 1994) if

$$X \leq_L Y \Rightarrow X \leq_{hr} \Rightarrow X \leq_m Y$$

$$\Downarrow$$

$$X \leq_{st} Y$$

- Hazard rate order $(X \leq_{hr} Y)$ if $P_x(x) \geq P_y(x)$ for all x
- Stochastic order $(X \leq_{st} Y)$ if $G_X(x) \geq G_Y(x)$ for all x
- Mean residual life order $(X \leq_m Y)$ if $m_X(x) \leq m_Y(x)$ for all x
- Likelihood ratio order $(X \leq_L Y)$ if $\frac{P_x(x)}{P_y(x)}$ decreases in x

Theorem 4.1 *Let X and Y follow the AGUE distribution with λ_1 and λ_2 respectively. If $\lambda_1 \geq \lambda_2$, then $X \leq_L Y$. Hence $X \leq_{hr} Y$, $X \leq_m Y$ and $X \leq_{st} Y$.*

Proof

The AGUE distribution will be ordered based on the strongest likelihood ratio ordering as established in Shaked (1994).

$$\frac{P_x(x;\lambda_1)}{P_y(x;\lambda_2)} = \frac{\frac{\lambda_1^3 exp(-\lambda_1 x)\left[\lambda_1^4+4x^2\right]}{(\lambda_1^6+8)}}{\frac{\lambda_2 exp(-\lambda_2 x)\left[\lambda_2^4+4x^2\right]}{(\lambda_2^6+8)}}. \quad (9)$$

$$\frac{d}{dx}\log\frac{P_x(x;\lambda_1)}{P_y(x;\lambda_2)} = \frac{-2(\lambda_1^4-\lambda_2^4)}{(\lambda_1^4+4x^2)(\lambda_2^4+4x^2)} - (\lambda_1-\lambda_2). \quad (10)$$

Thus, for $\lambda_1 > \lambda_2$, $\frac{d}{dx}\log\frac{P_x(x;\lambda_1)}{P_y(x;\lambda_2)} < 0 \Rightarrow X \leq_L Y$, for $\lambda_1 < \lambda_2$, $\frac{d}{dx}\log\frac{P_x(x;\lambda_1)}{P_y(x;\lambda_2)} > 0$ and if $\lambda_1 = \lambda_2$, $\frac{d}{dx}\log\frac{P_x(x;\lambda_1)}{P_y(x;\lambda_2)} = 0$. Therefore $X \leq_{hr} Y$,$X \leq_m Y$ and $X \leq_{st} Y$.

5. Parameter estimation of the AGUE distribution

Let $x_1, x_2, x_3, \cdots, x_k$ be a random sample of size k observations sampled from the AGUE distribution. Then, for $\bar{x}$, the sample mean, the log-likelihood function of the AGUE distribution can be derived as

$$InL = 3kIn\lambda - kIn(\lambda^6 + 8) + \sum_{m=1}^{k} In(\lambda^6 + 4x_m^2) - k\lambda \sum_{m=1}^{k} x_m. \tag{11}$$

Thus, taking the partial derivative, we have

$$\frac{dInL}{d\lambda} = \frac{3k}{\lambda} - \frac{k6\lambda^5}{(\lambda^6 + 8)} + \sum_{m=1}^{k} \left[\frac{6\lambda^5}{(\lambda^6 + 4x_m^2)} \right] - k\bar{x}. \tag{12}$$

The estimate of the parameter λ can be obtained by equating to zero. The Newton-Raphson algorithm Software like R, MATLAB, MAPLE and others are used to obtain the estimate.

5.1. The AGUE distribution regression model

In this section, we introduce the regression model for the AGUE distribution. In the literature, numerous researchers have introduced regression form of probability model intending to improve on the model flexibility and also make predictions easier using such a regression model such as the Bivariate exponentiated-exponential geometric regression model (Famoye, 2019), Exponentiated-exponential geometric regression model (Famoye and Carl, 2016), Bivariate Weibull regression model based on censored samples (Hanagal, 2006), Transmuted Burr Type X distribution regression model (Khan, King, and Hudson, 2019), and Transmuted Log-logistic regression model (Granzotto and Louzada, 2015), among others.

However, the Granzotto and Louzada (2015) approach is adopted for the AGUE model.

Consider Z to be a random variable with pdf of the AGUE distribution and $g(x) = \lambda$ as a parameter depending on the covariate vector $X = (1, x_m, \cdots, x_k)^T$, where $m = 1, 2, \cdots, k$ and $g(x) = \lambda = \theta_0 + \theta_m x_m + \cdots + \theta_k x_k$. Hence, the pdf of the AGUE distribution can be redefined as

$$P(z|g(x)) = \frac{(g(x))^3 \left[(g(x))^4 + 4z^2 \right] exp(-g(x)z)}{(g(x))^6 + 8}, \tag{13}$$

where $g(x)$ is a regression model. The corresponding survival $S(z|g(x))$ and hazard $h(z|g(x))$ rate functions at period z of (13) can be written as

$$S(z|g(x)) = \frac{\left[(g(x))^6 + 4(g(x)z)^2 + 8(g(x)z + 1) \right] exp(-g(x)z)}{(g(x))^6 + 8}. \tag{14}$$

$$h(z|g(x)) = \frac{(g(x))^3 \left[(g(x))^4 + 4z^2 \right]}{(g(x))^6 + 4(g(x)z)^2 + 8(g(x)z + 1)}. \tag{15}$$

Let $p_m, \cdots, p_k$ be a sample of size from the AGUE distribution, and $X_m = (1, x_{1m}, \cdots, x_{km})^T$, be an m^{th} vector of covariates, $m = 0, 1, \cdots, k$ and $g(x) = \lambda = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$. Also, for convenience notation, let set $x_0 = 1$, then $g(x) = \theta X^T$, where $\theta = (\theta_0, \theta_1, \cdots, \theta_k) \in \mathbb{R}^{1\times(n+1)}$ and

$$X = \begin{pmatrix} x_0 \\ x_1 \\ . \\ . \\ . \\ x_k \end{pmatrix} \in \mathbb{R}^{1\times(n+1)}.$$

Thus, the log-likelihood function can be written as

$$L = InL(g(x)|p,x) = 3k\sum_{m=0}^{k} In\big((g(x))^6 + 8\big) + \sum_{m=0}^{k}\big((g(x))^6 + 4p_m^2\big) - k\sum_{m=0}^{k} g(x)p_m. \quad (16)$$

The maximum likelihood estimates of the parameters of $g(x)$, which maximizes (16), must satisfy the equations

$$\frac{dL}{d\theta_0} = \frac{3k}{\sum_{m=0}^{k} g(x)} - \frac{6k\sum_{m=0}^{k}(g(x))^5}{\sum_{m=0}^{k}\big[(g(x))^6 + 8\big]} + 6\sum_{m=0}^{k}(g(x))^5 - k\sum_{m=0}^{k} y_m = 0.$$

$$\frac{dL}{d\theta_m} = \frac{3k\sum_{m=0}^{k} x_m}{\sum_{m=0}^{k} g(x)} - \frac{6k\sum_{m=0}^{k} x_m(g(x))^5}{\sum_{m=0}^{k}\big[(g(x))^6 + 8\big]} + 6\sum_{m=0}^{k} x_m(g(x))^5 - k\sum_{m=0}^{k} x_m p_m = 0.$$

6. Real-life data applications

The numerical applications of the one-parameter AGUE distribution are demonstrated using two data sets.

Data set I is a data set report consisting of 63 observations of the strengths of 1.5cm glass fibers. The data set has previously been analyzed in Sharma et al.(2016), Oguntunde et al.(2017), Abdal-hameed et al. (2018), Eghwerido et al. (2021), Oguntunde et al.(2018), Eghwerido and Agu (2021), Eghwerido, Agu and Ibidoja (2021a) and Khaleel, Al-Noor and Abdal-Hameed (2020).

Data set II is a data set of about 346 nicotine measurements collected by the Federal Trade Commission. [http://www.ftc.gov/ reports/tobacco or https: // pw1.netcom.com/ rdavis2/ smoke. html.] used in Handique and Chakraborty (2016).

Tables 3a, 3b, 4a, and 4b present the parameter estimate and values of the test statistics for the fitted models on the data sets. In addition, the model parameter (Par.) and their corresponding Log-likelihood (LL), standard errors (Str. Error), and confidence intervals (CI) are presented. R-programming was used to obtain the results. However, in Tables 3a, 3b, 4a, and 4b, the AGUE model test statistics are the lowest among all fitted models for the data sets. Thus, the AGUE model is chosen as the best model among others for these data sets.

Figures 6, 7, 8, and 9 show the plots of the empirical estimated densities and density of the models considered for the data sets.

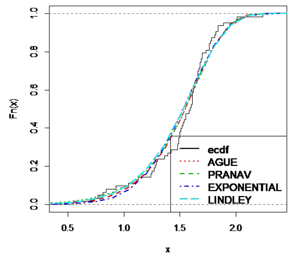

Figure 6: The plots of the estimated ecdf for the AGUE distribution for data set I

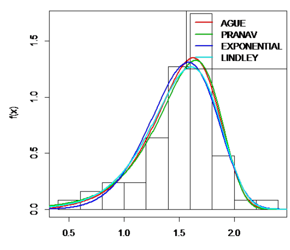

Figure 7: The plots of the estimated density of the AGUE distribution for data set

Table 3a. Parameter estimates of the strengths of 1.5 cm glass fibers data.

Model	Par.	Est.	Str. Error	LL	CI (95%)	
					Upper	Lower
AGUE	$\hat{\lambda}$	4.590	0.225	-237.634	5.031	4.149
Exponential	$\hat{\lambda}$	0.664	0.084	-88.830	0.829	0.499
Pranav	$\hat{\lambda}$	1.561	0.079	-90.481	1.716	1.406
Lindley	$\hat{\lambda}$	0.996	0.095	-81.278	1.182	0.800

Table 3b. The test statistics values of the strengths of 1.5 cm glass fibers data.

Model	AIC	CAIC	BIC	HQIC
AGUE	-473.268	-473.2024	-471.125	-472.425
Exponential	-179.661	-179.726	-181.804	-180.504
Pranav	-182.963	-183.028	-185.106	-183.806
Lindley	-164.557	-164.623	-166.700	-165.390

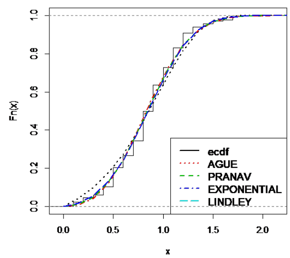

Figure 8: The plots of the estimated ecdf for the AGUE distribution for data set II

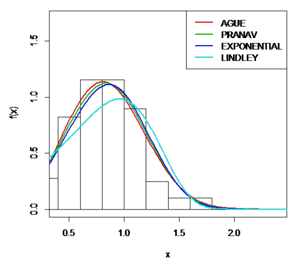

Figure 9: The plots of the estimated density of AGUE distribution for data set II

Table 4a. Parameter estimates of several brands of cigarettes data.

Model	Par.	Est.	Str. Error	LL	CI (95%)	
					Upper	Lower
AGUE	$\hat{\lambda}$	8.207	0.167	-2677.47	8.534	7.870
Exponential	$\hat{\lambda}$	1.173	0.063	-290.826	1.296	1.049
Pranav	$\hat{\lambda}$	2.056	0.052	-345.079	2.157	1.955
Lindley	$\hat{\lambda}$	1.620	0.068	-269.796	1.755	1.486

Table 4b. The test statistics values of several brands of cigarettes data.

Model	AIC	CAIC	BIC	HQIC
AGUE	-5352.941	-5352.929	-5349.094	-5351.409
Exponential	-583.651	-583.663	-587.498	-585.183
Pranav	-692.157	-692.169	-696.004	-693.689
Lindley	-541.593	-541.605	-545.439	-543.125

7. Conclusions

We introduced one-parameter distribution called the AGUE distribution with its mathematical representation and parameter estimation in this study. The regression model and basic statistical properties such as the index of dispersion and others were explored. The

AGUE parameter is estimated using the method of maximum likelihood estimation. The lifetime applications of the AGUE distribution was illustrated using two-lifetime data sets. The characteristic of the introduced model for larger sample size is examined via simulation study. The AGUE distribution has the lowest value of test statistics. Thus, it provides the best fit and more flexible than Pranav, exponential, and Lindley distributions for the data sets. Ultimately, the AGUE distribution can serve as an alternative model to Pranav, exponential, and Lindley distributions in the literature. A further research question is how the applications of the regression model of the AGUE distribution can be explored on real lifetime data.

7.1. Appendix: Probability density function

Let us write as Mood, Graybill and Boes (1974) stated that any function $P(.)$ is defined to be a pdf if and only if the following conditions are satisfied

1. $P(x) \geq 0$ for all x and
2. $\lim_{c\to\infty}\int_{-\infty}^{c} P(x)dx = 1$.

It is easy to see that the first property is satisfied for all $x > 0$. The second property is shown as follows. Firstly,

$$\begin{aligned}\lim_{c\to\infty}\int_0^c P(x)dx &= \int_0^\infty \frac{\lambda^3}{\lambda^6+8}\left[\lambda^4+4x^2\right]exp(-\lambda x)dx \\ &= \frac{\lambda^3}{\lambda^6+8}\lim_{c\to\infty}\Big[\int_0^c \lambda^4 exp(-\lambda x)dx + \int_0^c 4x^2 exp(-\lambda x)dx\Big].\end{aligned}$$

By performing integration by parts, we obtained

$$\frac{\lambda^3}{\lambda^6+8}\left[\lambda^3+\frac{8}{\lambda^3}\right] = 1.$$

Therefore, equation (1) is a pdf.

Acknowledgements

Thanks to the anonymous reviewers and the Editor in chief.

Conflıct of ınterest

The authors state that there is no conflict of interest related to this study.

References

Abdal-hameed, M. K., Khaleel, M. A., Abdullah, Z. M., Oguntunde, P. E., Adejumo, A. O., (2018). Parameter estimation and reliability, hazard functions of Gompertz Burr Type XII distribution. *Tikrit Journal for Administration and Economics Sciences*, 1(41-2), pp. 381–400.

Agu, F. I., Onwukwe, C. E., (2019). Modified Laplace Distribution, Its Statistical Properties and Applications. *Asian Journal of Probability and Statistics*, pp. 1–14.

Agu, F. I, Francis, R. E., (2018). Comparison of goodness of fit tests for normal distribution. *Asian Journal of Probability and Statistics*, pp. 1–32.

Bakouch, H. S., Ristić, M. M., Asgharzadeh, A., Esmaily, L., Al-Zahrani, B. M., (2012). An exponentiated exponential binomial distribution with application. *Statistics and Probability Letters*, 82(6), pp. 1067–1081.

Eghwerido, J. T., Nzei, L. C., Agu, F. I., (2020). The Alpha Power Gompertz Distribution: Characterization, Properties, and Applications.*Sankhya A - The Indian Journal of Statistics*, https://doi.org/ 10.1007/s13171-020-00198-0.

Eghwerido, J. T., Oguntunde, P. E., Agu, F. I., (2020). The Alpha Power Marshall-Olkin-G Distribution: Properties, and Applications. *Sankhya A - The Indian Journal of Statistics*, https://doi.org/10.1007/s13171-020-00235-y.

Eghwerido, J. T., Agu, F. I., (2021). The Shifted Gompertz-G family of Distributions: Properties and Applications. *Mathematica Slovaca*, Article in the press.

Eghwerido, J. T., Agu, F. I. Ibidoja, J. O., (2021a). The Shifted Exponential-G family of distributions: properties and applications. *Journal of Statistics and Management System*, https://doi.org/10.1080/09720510.2021.1874130.

Famoye, F., (2019). Bivariate exponentiated-exponential geometric regression model. Statistica Neerlandica, 73(3), 434–450.

Famoye, F., Carl Lee., (2017). Exponentiated-exponential geometric regression model. *Journal of Applied Statistics*, 44(16), pp. 2963–2977.

Granzotto, D. C. T., Louzada, F., (2015). The transmuted log-logistic distribution: modeling, inference, and an application to a polled tabapua race time up to first calving data. *Communications in Statistics-Theory and Methods*, 44(16), pp. 3387–3402.

Gómez-Déniz, E., Calderín-Ojeda, E., (2011). The discrete Lindley distribution: properties and applications. *Journal of Statistical Computation and Simulation*, 81(11), pp. 1405-1416.

Ghitany, M. E., Atieh, B., Nadarajah, S., (2008). Lindley distribution and its application. *Mathematics and computers in simulation*, 78(4), pp. 493–506.

Ghitany, M. E., Al-Mutairi, D. K., Balakrishnan, N., Al-Enezi, L. J., (2013). Power Lindley distribution and associated inference. *Computational Statistics and Data Analysis*, 64, pp. 20–33.

Hanagal, D. D., (2006). Bivariate Weibull regression model based on censored samples. *Statistical Papers*, 47(1), pp. 137–147.

Handique, L., Chakraborty, S., (2016). Beta generated Kumaraswamy-G and other new families of distributions. *arXiv* preprint arXiv:1603.00634.

Khaleel, M. A., Al-Noor, N. H., Abdal-Hameed, M. K. Marshall Olkin exponential Gompertz distribution: Properties and applications. *Periodicals of Engineering and Natural Sciences*, 8(1), pp. 298–312.

Khan, M. S., King, R., Hudson, I. L., (2020). Transmuted Burr Type X Distribution with Covariates Regression Modeling to Analyze Reliability Data. *American Journal of Mathematical and Management Sciences*, 39(2), pp. 99–121.

Lindley D. V., (1958). Fiducial distributions and Bayes' theorem. *Journal of the Royal Statistical Society, Series B*, 20, 1(1), pp. 102–107.

Merovci, F., (2013). Transmuted Rayleigh distribution. *Austrian Journal of Statistics*, 42(1), pp. 21 –31.

Mood AM, Graybill FA, Boes DC., (1974). Introduction to the theory of statistics, 3rd edn. McGraw Hill, *New York*

Nadarajah, S., Haghighi, F., (2011). An extension of the exponential distribution. *Statistics*, 45(6), pp. 543–558.

Odom, C. C., Ijomah, M. A., (2019). Odoma Distribution and Its Application. *Asian Journal of Probability and Statistics*, pp. 1-11.

Oguntunde, P. E., Khaleel, M. A., Ahmed, M. T., Adejumo, A. O., Odetunmibi, O. A., (2017). A New Generalization of the Lomax Distribution with Increasing, Decreasing, and Constant Failure Rate. Modelling and Simulation in Engineering, 2017.

Oguntunde, P. E., Khaleel, M. A., Adejumo, A. O., Okagbue, H. I., (2018). A study of an extension of the exponential distribution using logistic-x family of distributions. *International Journal of Engineering and Technology*, 7(4), pp. 5467–5471.

Oluyede, B. O., Yang, T., (2014). Generalizations of the inverse Weibull and related distributions with applications. *Electronic Journal of Applied Statistical Analysis*, 7(1), pp. 94.

Sharma, V. K., Singh, S. K., Singh, U., Merovci, F., (2016). The generalized inverse Lindley distribution: A new inverse statistical model for the study of upside-down bathtub data. *Communications in Statistics-Theory and Methods*, 45(19), pp. 5709–5729.

Shaked, M., Shanthikumar, J. G., (1994). Stochastic Orders and Their Applications, Academic Press, *New York*.

Shanker, R., (2015a). Akash distribution and its applications. *International Journal of Probability and Statistics*, 4(3), pp. 65–75.

Shanker, R., (2016). Aradhana distribution and its Applications. *International Journal of Statistics and Applications*, 6(1), pp. 23-34.

Shanker, R., Shukla, K. K., (2017). Ishita distribution and its applications. *Biometrics and Biostatistics International Journal*, 5(2), pp. 1–9.

Shanker, R., (2016). Sujatha distribution and its Applications. *Statistics in Transition. New Series*, 17(3), pp. 391–410.

Shanker, R., Mishra, A., (2013). On a Size-Biased Quasi Poisson-Lindley Distribution. *International Journal of probability and Statistics*, 2(2), pp. 28–34.

Shanker, R., Amanuel, A. G., (2013). A new quasi Lindley distribution. *International Journal of Statistics and systems*, 8(2), pp. 143-156.

Shukla K. K., (2018). Pranav distribution with properties and its applications. *Biom Biostat Int J*, 7(3), pp. 244–254.

Warahena-Liyanage, G., Pararai, M., (2014). A generalized power Lindley distribution with applications. *Asian journal of mathematics and applications*, 2014, pp. 1–23.

Zakerzadeh, H., Dolati, A., (2009). Generalized Lindley distribution. *Scientific Information Database*, 3(2), pp. 13–25.